

REPRESENTATION THEORY OF LIFTINGS OF QUANTUM PLANES

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ABSTRACT. We determine the regular representations, Gabriel quivers and representation type of all liftings of two-dimensional quantum linear spaces.

INTRODUCTION

Liftings of quantum linear spaces were constructed and completely described in [3], and independently in [6]. These Hopf algebras belong to the class of pointed finite-dimensional Hopf algebras with an abelian group of group-likes and are, arguably, the simplest Hopf algebras constructed by the lifting method of N. Andruskiewitsch and H.-J. Schneider (see e.g. [4]). When the dimension of a quantum linear space is two, we colloquially refer to them as quantum planes. In this article we systematically study the representation theory of liftings of quantum planes. The group of group-likes is arbitrary finite abelian in our construction, and the base field \mathbb{k} is algebraically closed of characteristic zero. Liftings of quantum planes are generated as algebras by two skew-primitive elements along with the group-likes G . The skew-primitives generalize the generators e, f in the restricted enveloping algebra of $\mathfrak{sl}(2)$ and quantum analogs at roots of unity. If there is a nontrivial commutation relation between the skew-primitives we say that the lifting is linked.

Let us summarize some existing related work. The simple representations of liftings of quantum planes were treated in [1, 2] where simple modules are described or reduced to known theory in most cases. The representation theory of various versions of quantized restricted enveloping algebras was studied in [7, 23, 21] using a variety of techniques. In [11] representations of quantum doubles of generalized Taft algebras were examined. The simple and projective modules were explicitly constructed within the regular representation, extending methods from [21]. These Hopf algebras are examples of liftings of quantum planes in the linked nilpotent case. In [13] rank one Hopf algebras were constructed and the structure of the regular representation of their doubles was obtained. These Hopf algebras are sometimes liftings of quantum planes; but this is not always the case as sometimes they are not even pointed.

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In this article we explicitly describe the simples, projectives, blocks, and Gabriel quiver in all cases. This enables us to determine the representation types of the blocks. We make use of certain central idempotents that are constructed from certain equivalence classes of characters of G to reduce to the corresponding two-sided summands, which we call *class subalgebras*. These idempotents were also used in [1, 2]. The class subalgebras we encounter are generated by three elements, one of which is a unit, and the other two elements are the images of skew-primitives. Our analysis breaks into the consideration of summands where both, just one, or neither of the generators is nilpotent. We shall refer to these cases as *nilpotent*, *seminilpotent* and *unipotent*, respectively. Another division is into the classes of linked and unlinked liftings, resulting in six distinct cases altogether.

Let us give an outline of the results. In the unlinked nonnilpotent cases, the class subalgebras are the blocks, and are either Nakayama algebras, or skew group algebras over truncated polynomial rings. The Nakayama algebras are the simplest algebras of finite representation type, while the truncated polynomial rings are among simplest of the algebras of wild and tame representation type (depending on the degree of truncation). In the linked nonnilpotent cases the class subalgebras are direct sums matrix algebras over a Nakayama algebras and are thus of finite representation type.

In the remaining linked nilpotent case the nonsimple blocks are special biserial algebras and therefore of tame representation type. The algebras in this case generalize the quantized restricted enveloping algebra of $\mathfrak{sl}(2)$. The basic algebras and quivers that occur here also arose in the study of Hopf algebras in [11], and we can apply their results to this case. Here we use an alternative less explicit but much simpler determination of the simple and projective representations using analogs of “baby Verma modules” in the more general setting of seminilpotent and nilpotent liftings of quantum planes. We show that linked liftings are symmetric algebras and we provide an analog of the Casimir element. These tools allow a concise and efficient determination of the socle and Loewy series structure of the projective indecomposables, and the minimal central idempotents of nonsimple blocks.

A more detailed description of material by sections is as follows. In section 1 we review the ideas leading up to the construction of liftings of quantum linear spaces. We define types of liftings and discuss some auxiliary facts about duality for finite abelian groups and group algebras used in the sequel.

In section 2 we introduce the general theory of liftings of quantum planes. We define class idempotents, class subalgebras and compute bases for them. A quick argument proving Lemma 2.2 shows that the class idempotents are central. In §2.1 we give generators and relations for liftings and comment on the existence of unlinked or linked data.

In section 3 we determine the structure of simples, projectives, Gabriel quiver and blocks for unlinked liftings. The nilpotent case is handled in Theorem 3.1, where the class subalgebras are the blocks and are essentially

skew group rings over truncated polynomial rings. The seminilpotent case is described in Theorem 3.2 where the class subalgebras either resemble the nilpotent case or are nonbasic Nakayama algebras. In this case the Gabriel quiver has vertices corresponding to certain cosets of a certain subgroup of the character group. In the unipotent case, we show in Theorem 3.3 that the class subalgebras are semisimple, and we explicitly construct the simple modules. We also give a criterion for the isomorphism of the simples in Theorem 3.4.

We turn to the linked liftings in section 4, where we find an analog of the Casimir element from classical Lie theory, which is central by Lemma 4.1. In Proposition 4.2 we show that linked liftings are symmetric algebras by directly showing unimodularity and that the square of the antipode is inner. In §4.2 we study the image of the Casimir element C in each class subalgebra and compute the minimal polynomials. In the unipotent case we quickly find in Theorem 4.5 that each class subalgebra is isomorphic to a matrix algebra over the subalgebra generated by C . As a result, each class subalgebra is a direct sum of matrix rings over local rings. In the seminilpotent case, addressed in Theorem 4.7 where the blocks are matrix rings over the base field or over the truncated polynomial algebra $\mathbb{k}[v]/(v^2)$. The precise decompositions are expressed in a number of cases. For use in the seminilpotent and nilpotent cases, we introduce analogs of standard cyclic modules in §4.5, where they are shown to be simple in Proposition 4.8 in the seminilpotent case. Finally we study the lengthier nilpotent case in §4.6. Here the standard cyclic modules may no longer be simple, but have a unique maximal submodule. Paralleling Lie theory, we obtain precise results concerning the simple quotients. As a first step in Theorem 4.13, we dispose of the generic case where the class subalgebra is semisimple. We then look at the complementary case and we give detailed results about the Loewy factors of the standard cyclic modules. This enables the determination of the structure of the projective indecomposable modules in Theorem 4.18, along with a presentation of the basic algebras of blocks by quivers with relations in Theorem 4.20. We close by explicitly giving embeddings of the projective indecomposables into nonsimple blocks.

1. PRELIMINARIES

Notation: G a finite abelian group

\widehat{G} the character group of G

\mathbb{k} an algebraically closed field

$a_1, \dots, a_n \in G$

$\chi_1, \dots, \chi_n \in \widehat{G}$

$q_{ij} = \chi_j(a_i), \quad q_i = \chi_i(a_i)$

For $g \in G$ $|g|$ denotes the order of g . In particular, for $q \in \mathbb{k}^\bullet$ $|q|$ is the order of q .

For $\kappa \in \mathbb{k}$ and $N \in \mathbb{N}$, $R_{\kappa, N} = \{\gamma \in \mathbb{k} \mid \gamma^N = \kappa\}$

1.1. Liftings of Quantum Linear Spaces. Recall [22, 19, 4] that the category ${}^G\mathcal{YD}$ consists of $\mathbb{k}G$ -modules and $\mathbb{k}G$ -comodules V such that the G -grading

$$V = \bigoplus V_g \text{ where } V_g = \{v \in V | \rho(v) = g \otimes v\}$$

where $\rho : V \rightarrow \mathbb{k}G \otimes V$ is the comodule structure map, satisfies

$$h.V_g \subset V_{hgh^{-1}} \text{ for every } h, g \in G.$$

Since G is abelian $h.V_g = V_g$ and V has a basis, say, $\{v_i | i \in I\}$ of G and \widehat{G} -eigenvectors indexed by a set I . In other words,

$$(1.1) \quad g.v_i = \chi_i(g)v_i$$

$$(1.2) \quad \rho(v_i) = a_i \otimes v_i$$

for some $a_i \in G$ and $\chi_i \in \widehat{G}$, and for all $g \in G$, $i \in I$.

We say that an element $v \in V$ is *bihomogeneous* of degree (a, χ) if the equations (1.1) and (1.2) hold for v, g and χ and we write $a = g_v, \chi = \chi_v$.

Definition 1.1. ([3]) An n -dimensional Yetter-Drinfel'd module V is called a quantum linear space if

$$\chi_i(a_j)\chi_j(a_i) = 1 \text{ for all } i \neq j$$

in some bihomogeneous basis for V .

We review a construction of the Nichols algebra $B(V)$ associated to the quantum linear space V (cf. [3, Lemma 3.4]). As an algebra $B(V)$ is defined via the relations

$$\begin{aligned} v_i v_j &= \chi_j(a_i) v_j v_i \\ v^{n_i} &= 0 \end{aligned}$$

for all $i \neq j$ and $n_i = |q_i|$. One can see immediately that the set $\{v_1^{i_1} \cdots v_n^{i_n}\}$ is a basis of $B(V)$. The action and coaction of G on V both extend uniquely to $B(V)$ by requiring $B(V)$ to be an algebra in ${}^G\mathcal{YD}$. Explicitly

$$\begin{aligned} g.v_1^{i_1} \cdots v_n^{i_n} &= \chi_1^{i_1} \cdots \chi_n^{i_n}(g) v_1^{i_1} \cdots v_n^{i_n} \\ \rho(v_1^{i_1} \cdots v_n^{i_n}) &= a_1^{i_1} \cdots a_n^{i_n} \otimes v_1^{i_1} \cdots v_n^{i_n}. \end{aligned}$$

Note that every monomial $v_1^{i_1} \cdots v_n^{i_n}$ is bihomogeneous.

The coalgebra structure maps δ, ϵ are defined by first setting

$$\epsilon(v_1^{i_1} \cdots v_n^{i_n}) = \delta_{0, i_1} \cdots \delta_{0, i_n}$$

and for $\delta : B(V) \rightarrow B(V) \otimes B(V)$ is given by

$$\delta(v_i) = v_i \otimes 1 + 1 \otimes v_i$$

on generators and extends to $B(V)$ as follows. Following Lusztig [14] or [4] we give $B(V) \otimes B(V)$ a new multiplication by setting

$$(r \otimes s)(t \otimes u) = \chi_t(g_s) r t \otimes s u$$

for monomials r, s, t, u , and denote this algebra by $B(V) \underline{\otimes} B(V)$. We also endow $B(V) \otimes B(V)$ with the usual tensor product module and comodule structures over $\mathbb{k}G$. A tedious, but straightforward calculation shows that $B(V) \underline{\otimes} B(V)$ is an algebra in ${}^G\mathcal{YD}$, δ is well-defined and δ, ϵ are G -linear and G -colinear algebra maps. The above mentioned properties amount to saying that $B(V)$ is a bialgebra in ${}^G\mathcal{YD}$. We turn $B(V)$ into a Hopf algebra in ${}^G\mathcal{YD}$ by defining an antipode \mathcal{S} as the linear map

$$\mathcal{S}(v_1^{i_1} \cdots v_n^{i_n}) = (-1)^{\sum i_j} \prod_{j=1}^n q_j^{\binom{i_j}{2}} v_1^{i_1} \cdots v_n^{i_n}.$$

We can use now a result of Radford [20], recast by Majid [15] in categorical terms, to the effect that the biproduct or bosonization $B(V) \# \mathbb{k}G$ is a Hopf algebra. The algebra $B(V) \# \mathbb{k}G$ is known as the *trivial lifting* of V .

Definition 1.2. A Hopf algebra H is a lifting of V if the graded Hopf algebra associated to the coradical filtration of H is isomorphic to $B(V) \# \mathbb{k}G$.

Next we derive a general property of liftings.

Proposition 1.3. *Every lifting of V has a structure of left-left Yetter-Drinfel'd module.*

PROOF: Let H be a lifting of V . By [3] $H_0 = \mathbb{k}G$ hence H is naturally a G -module under the action of G by conjugation, $h \mapsto g.h := ghg^{-1}$ for every $h \in H, g \in G$. Moreover, H is generated by G and skew-primitives x_i satisfying $g.x_i = \chi_i(g)x_i$. By [3, 5.2] the set $\{gx_1^{i_1} \cdots x_n^{i_n} | g \in G, 0 \leq i_j < n_j\}$ is a basis for H . Let I be the span of all $gx_1^{i_1} \cdots x_n^{i_n}$ with $\sum i_j > 0$. By the quantum binomial formula [12] one can see readily that I is a coideal and it complements H_0 . Therefore we have a coalgebra projection $\pi : H \rightarrow \mathbb{k}G$. Then $\rho := (\pi \otimes \text{id})\Delta : H \rightarrow \mathbb{k}G \otimes H$ equips H with a $\mathbb{k}G$ -comodule structure. Further, every $u = gx_1^{i_1} \cdots x_n^{i_n}$ is bihomogeneous of degree $(ga_1^{i_1} \cdots a_n^{i_n}, \chi_1^{i_1} \cdots \chi_n^{i_n})$. Therefore for every $h \in G$

$$\rho(h.u) = \chi_1^{i_1} \cdots \chi_n^{i_n}(h)\rho(u) = ga_1^{i_1} \cdots a_n^{i_n} \otimes h.u$$

and the proof is complete. \square

We apply the previous proposition and consider the braided commutator $[\cdot, \cdot]_c : H \otimes H \rightarrow H$ corresponding to the braiding $c : H \otimes H \rightarrow H \otimes H$ arising from the Yetter-Drinfel'd module structure on H [4]. This commutator is given by $[a, b]_c = \mu(\text{id} - c)(a \otimes b)$ for $a, b \in H$ where μ is the multiplication in H . When a, b are bihomogeneous we have

$$[a, b]_c = ab - \chi_b(g_a)ba.$$

We shall later have use of the following braided commutator rules. Let x, y be bihomogeneous elements of degrees (a, χ) and (b, χ^{-1}) , respectively, satisfying

$$(1.3) \quad [x, y]_c = ab - 1 \quad \text{and} \quad \chi(a) = \chi(b).$$

Proposition 1.4. *Let H be a lifting of V . Suppose $u, v \in H$ are bihomogeneous and x, y are as above. Then*

- (a) $[x, uv]_c = [x, u]_c v + \chi_u(a) u[x, v]_c$
- (b) Let $q = \chi^{-1}(a)$. For every $s \geq 1$
 - (i) $[x, y^s]_c = (s)_q y^{s-1} (q^{s-1} ab - 1)$
 - (ii) $[y, x^s]_c = q^{-s} (s)_q x^{s-1} (q^{-(s-1)} ab - 1)$.

PROOF: (a) is seen by a direct inspection.

(b) (i). The formula holds for $s = 1$ by 1.3. We induct on s assuming it holds for the given s . By part (a) we calculate

$$[x, y^{s+1}]_c = [x, y^s \cdot y]_c = (s)_q y^{s-1} (q^{s-1} ab - 1) y + q^s y^s (ab - 1)$$

because y^s is a G -eigenvector of weight χ^{-s} . Since $aby = q^2 yab$ the first term on the right is $(s)_q y^s (q^{s+1} ab - 1)$. Therefore the right hand side equals

$$\gamma y^s ((q^s + (s)_q q^{s+1}) ab - (q^s + (s)_q)) = \gamma y^s (s+1)_q (q^s ab - 1)$$

the last equality holds by the identities $1 + (s)_q q = (s+1)_q$ and $q^s + (s)_q = (s+1)_q$.

Part (b)(ii) is proven similarly using the relation $yx - q^{-1}xy = -q^{-1}(ab - 1)$ and $(s)_{q^{-1}} = q^{-s+1}(s)_q$. \square

1.2. Duality for abelian groups and group algebras. We collect some basic facts that will be used freely henceforth.

Let G be a finite abelian group and its dual group \widehat{G} as before. Then there is an inclusion-reversing correspondence between the subgroups of G and the subgroups of \widehat{G} . The correspondence takes a subgroup $H \subset G$ to

$$H^\perp = \{\lambda \in \widehat{G} \mid \lambda(h) = 1 \text{ for all } h \in H\}.$$

The inverse is defined similarly using the identification $\widehat{\widehat{G}} = G$ and we note that $H^{\perp\perp} = H$. We have $(H \cap K)^\perp = H^\perp + K^\perp$ and $(H + K)^\perp = H^\perp \cap K^\perp$ [16].

Now considering the Hopf group-algebra $\mathbb{k}G$ we extend the natural pairing

$$G \times \widehat{G} \rightarrow \mathbb{k}^\bullet, \langle g, \lambda \rangle \rightarrow \lambda(g)$$

to a bilinear pairing

$$\mathbb{k}G \times \mathbb{k}\widehat{G} \rightarrow \mathbb{k}, \langle \sum r_i g_i, \sum \rho_i \lambda_i \rangle \rightarrow \sum r_i \rho_i \lambda_i(g_i).$$

To every $\lambda \in \widehat{G}$ we associate a minimal idempotent

$$e_\lambda = \frac{1}{|G|} \sum_{g \in G} \lambda(g^{-1}) g$$

of $\mathbb{k}G$. The subspaces $\mathbb{k}e_\lambda$ afford a one-dimensional representation of G with character λ , i.e. $ge_\lambda = \lambda(g)e_\lambda$. Therefore

$$\begin{aligned} e_\lambda e_\mu &= \frac{1}{|G|} \left(\sum_{g \in G} \lambda(g^{-1}) \mu(g) \right) e_\mu \\ &= \mu(e_\lambda) e_\mu. \end{aligned}$$

By orthogonality of idempotents e_λ we have

$$(1.4) \quad \langle e_\lambda, \mu \rangle = \mu(e_\lambda) = \delta_{\lambda, \mu}.$$

Equality (1.4) can be interpreted as saying that basis $\{e_\lambda | \lambda \in \widehat{G}\}$ of $\mathbb{k}G$ is dual to the standard basis $\{\mu | \mu \in \widehat{G}\}$ of $\mathbb{k}\widehat{G}$. That is to say e_λ maps to the characteristic function p_λ under the Hopf algebra isomorphism $\mathbb{k}G \cong (\mathbb{k}\widehat{G})^*$. Therefore

$$\Delta(e_\lambda) = \sum_{\lambda = \mu\nu} e_\mu \otimes e_\nu.$$

2. GENERAL THEORY OF LIFTINGS

2.1. Types of Liftings. The general lifting H of V is given by a *lifting datum*

$$\mathcal{D} = \{G, a, b, \epsilon_1, \epsilon_2, \chi_1, \chi_2, \gamma\}$$

where $a, b \in G$, $\chi_i \in \widehat{G}$ and $\epsilon_i \in \{0, 1\}$, $\gamma \in \mathbb{k}$. H is generated by G and x, y subject to the relations of G and the following

$$(2.1) \quad x^{n_1} = \epsilon_1(a^{n_1} - 1)$$

$$(2.2) \quad x^{n_2} = \epsilon_2(b^{n_2} - 1)$$

$$(2.3) \quad gx = \chi_1(g)xg$$

$$(2.4) \quad gy = \chi_2(g)yg$$

$$(2.5) \quad xy - \chi_2(a)yx = \gamma(ab - 1)$$

The coalgebra structure of H is given by

$$(2.6) \quad \Delta(x) = a \otimes x + x \otimes 1$$

$$(2.7) \quad \Delta(y) = b \otimes y + y \otimes 1$$

$$(2.8) \quad \Delta(g) = g \otimes g$$

for all $g \in G$. The datum satisfies the conditions

$$(2.9) \quad 1 < n_1 = |\chi_1(a)|$$

$$(2.10) \quad 1 < n_2 = |\chi_2(b)|$$

$$(2.11) \quad \chi_1(b)\chi_2(a) = 1$$

$$(2.12) \quad \chi_i^{n_i} = \epsilon \text{ if } \epsilon_i = 1$$

$$(2.13) \quad \chi_1 \cdot \chi_2 = \epsilon \text{ and } ab \neq 1 \text{ if } \gamma \neq 0$$

We adopt the following terminology

- A lifting is *linked* if $\gamma \neq 0$

- A lifting is *nilpotent* if $\epsilon_i = 0$ for all i
- A lifting is *seminipotent* if $\epsilon_i = 0$ for exactly one i
- A lifting is *unipotent* if $\epsilon_i = 1$ for all i

We comment on existence of data. Assume G is a product of several cyclic groups, viz. $G = \langle g_1 \rangle \times \langle g_2 \rangle \times \cdots$. Suppose $|g_i| = m_i, i = 1, 2$ and let θ_i be a primitive root of 1 of order m_i . Define $\chi_i \in \widehat{G}, i = 1, 2$ by $\chi_i(g_j) = \delta_{ij}\theta_i$. The tuple $\{G, g_1, g_2, \chi_1, \chi_2, 0\}$ is a lifting datum. If G is a cyclic p -group generated by g , pick an integer s such that $s^2 \not\equiv 0 \pmod{|g|}$. Set $a = g, b = g^s$, and $\chi_1(a) = \theta, \chi_2(a) = \theta^{-s}$. Then $\{G, a, b, \chi_1, \chi_2, 0\}$ is a lifting datum.

Following [4] we call a datum *linkable* if $\chi_1\chi_2 = \epsilon$ and $ab \neq 1$. A linkable datum can be constructed as follows. Let $G \neq \mathbb{Z}_2$. Pick a subgroup L of G such that G/L is cyclic of order N , generated by $\bar{g} = gL$ with $g \in G$. Define a character $\phi : G/L \rightarrow \mathbb{k}^\bullet$ by sending \bar{g} to θ , where θ is a root of 1. Let χ be the pull-back of ϕ to G . We claim that there exists $a, b \in G$ such that $b = al, l \in L$ with $ab \neq 1$. If so, the tuple $\{G, a, b, \chi, \chi^{-1}, \gamma\}$ is a linkable datum for every $\gamma \in \mathbb{k}^\bullet$.

It remains to justify the claim about the elements a, b . If $L = 1$, then G is a cyclic group of order > 2 . Hence $g^2 \neq 1$. Thus $a = g = b$ will do.

Suppose $L \neq 1$. If $g^2 \notin L$, then $a = g, b = gl, l \in L$ will do. In case $g^2 \in L$, say, $g^2 = l_0$, then either $l_0 \neq 1$, and we can set $a = g = b$, or $g^2 = 1$ and then $a = g, b = gl, l \neq 1$ will do.

2.2. Class idempotents.

Definition 2.1. For each $\lambda \in \widehat{G}$ we associate the idempotent e_λ as above and let $X = \langle \chi_1, \chi_2 \rangle$. Let

$$e_{\lambda X} = \sum_{\mu \in \lambda X} e_\mu$$

for each coset λX .

For $h \in He_{\lambda X}$ we shall write \bar{h} for its image $he_{\lambda X} \in He_{\lambda X}$. Let T be a transversal for X^\perp in G . For each subgroup $L \subset G$, we shall write λ_L for the restriction of λ to L and put

$$e_{\lambda_L} = |L|^{-1} \sum_{g \in L} \lambda(g^{-1})g \in \mathbb{k}L.$$

Lemma 2.2. $e_{\lambda X}$ is a central idempotent in H for all $\lambda \in \widehat{G}$.

PROOF: Observe that

$$\begin{aligned} xe_\lambda &= |G|^{-1} \sum_{g \in G} \lambda(g^{-1})xg \\ &= (|G|^{-1} \sum_{g \in G} \lambda(g^{-1})\chi_1(g^{-1})g)x \\ &= e_{\lambda\chi_1}x \end{aligned}$$

and similarly $ye_\lambda = e_{\lambda\chi_2}y$. This yields the assertion. \square

Proposition 2.3. *The following sets*

- (a) $\{\bar{t}\bar{x}^{j_1}\bar{y}^{j_2} | 0 \leq j_i < n_i, t \in T\}$ and
- (b) $\{e_{\lambda\chi}\bar{x}^{j_1}\bar{y}^{j_2} | 0 \leq j_i < n_i, \chi \in X\}$

are bases of $He_{\lambda X}$.

PROOF: We have $e_\lambda g = \lambda(g)e_\lambda$ for all $\lambda \in X$ and $g \in G$. This yields $e_{\lambda X}g = \lambda(g)e_{\lambda X}$ if $g \in X^\perp$. It follows directly that the set in (a) in the statement spans $He_{\lambda X}$. The cardinality of the set in (a) is $[G : X^\perp]n_1n_2$, and since $[G : X^\perp] = |X|$, the set has cardinality $|X|n_1n_2$. Summing over a transversal for X in \widehat{G} gives a spanning set for $\oplus H_{\lambda X}$ having cardinality $[\widehat{G} : X]|X|n_1n_2 = |G|n_1n_2$. This is just the dimension of H , so our spanning set is a basis of H . This trivially implies that the spanning set in (a) is a basis for $He_{\lambda X}$.

The set in (b) is clearly a spanning set for $He_{\lambda X}$, and as in the proof of (a), it is a basis because it has the requisite cardinality $|X|n_1n_2$. \square

Lemma 2.4. $e_{\lambda X} = e_{\lambda_{X^\perp}}$

PROOF: Since $e_{\lambda_{X^\perp}} \in \mathbb{k}X^\perp \subset \mathbb{k}G$, we can express

$$e_{\lambda_{X^\perp}} = \sum_{\mu \in Y} e_\mu$$

in terms of the basis $\{e_\mu\}$ for $\mathbb{k}G$, indexed over some set $Y \subset \widehat{G}$. Multiplying this expression by $h \in X^\perp$ we conclude that

$$\lambda(h)e_{\lambda_{X^\perp}} = \sum_{\mu \in Y} \mu(h)e_\mu.$$

This implies that $\mu_{X^\perp} = \lambda_{X^\perp}$, i.e. $\mu \in \lambda X$. Therefore the number of μ 's occuring in the sum equals the dimension of $e_{\lambda_{X^\perp}}\mathbb{k}G$, which is $|G/X^\perp| = |X|$. It follows that the sum runs over all of λX , as desired. \square

3. REPRESENTATIONS OF UNLINKED LIFTINGS

3.1. Nilpotent unlinked liftings.

Theorem 3.1. *Let H be an unlinked lifting and suppose $\epsilon_1 = 0 = \epsilon_2$. Then*

- (a) *Every indecomposable projective module has the form $P_\lambda = He_\lambda$ for some $\lambda \in \widehat{G}$*
- (b) *The central idempotents of H are the e_λ indexed by the cosets $\lambda X \in \widehat{G}/X$.*
- (c) *The Gabriel quiver for each block $He_{\lambda X}$ has vertices corresponding to elements of λX and a pair of arrows*

$$\mu \rightarrow \mu\chi_1$$

$$\mu \rightarrow \mu\chi_2$$

for every $\mu \in \lambda X$.

- (d) H is of wild representation type, unless $n_1 = n_2 = 2$ in which case it is of tame representation type.

PROOF: Set $J = xH + yH$. By the hypothesis, J is a nilpotent ideal, and so is the Jacobson radical of H . Therefore H has $|G|$ indecomposable projective modules. Thus the projective modules He_λ , $\lambda \in \widehat{G}$ account for all of them. This proves (a).

We let $L_\lambda = He_\lambda / Je_\lambda$ denote the corresponding simple H -module. It is easy to see that for all $r \in \mathbb{N}$, $\lambda \in \widehat{G}$,

$$J^r e_\lambda = \sum_{i+j \geq r} \mathbb{k} x^i y^j e_\lambda,$$

whence each L_λ is one-dimensional, and $Je_\lambda / J^2 e_\lambda$ is two-dimensional, being spanned by the images of x and y . Note that the respective simple modules have weights $\lambda\chi_1$ and $\lambda\chi_2$. This says that $\text{Ext}_H^1(L_\lambda, L_{\lambda\chi_i})$, $i = 1, 2$ are one-dimensional. As $\lambda \in \widehat{G}$ is arbitrary, this accounts for the arrows labelled as in the statement. Thus the blocks are precisely indexed by the cosets as claimed. This completes the proof of (c).

Since H is a basic algebra, the central idempotents are exactly sum of the primitive idempotents corresponding to the vertices in each connected component of the quiver. This yields (b).

H is a skew group ring over the subalgebra generated by x and y . One can easily see that we can replace x by $a^{-1}x$ and assume that x and y commute. Thus H is a skew group ring over a truncated polynomial ring, say A . Since $|G|$ is invertible in \mathbb{k} , the arguments [5, 6.3] show that A and $H = AG$ have the same representation type. The assertion for the truncated polynomial rings follows from [9], whence (d). \square

3.2. Seminilpotent unlinked liftings. Next we describe indecomposable projective modules for unlinked data with exactly one $\epsilon_i = 0$. Let X_i denote the subgroup of \widehat{G} generated by χ_i , and put $X = X_1 X_2$ as before. We introduce another subgroup of \widehat{G} in this setting as follows. Let N denote the subgroup $\langle a^{n_1} \rangle^\perp$ of \widehat{G} . Note that $X \subset N$. This can be argued as follows. Since $\chi_1^{n_1} = \epsilon$ and $\chi_2(a) = \chi_1(b^{-1})$, we obtain $\chi_2(a^{n_1}) = 1$, which proves the inclusion.

We let N/X_1 denote the cosets modulo X_1 represented by elements of N as usual, and write $(\widehat{G} \setminus N)/X_1$ for the complementary set of cosets. Note that the disjoint sets N/X_1 and $(\widehat{G} \setminus N)/X_1$ are stable under the action by multiplication by elements of X .

Theorem 3.2. *Suppose H is an unlinked seminilpotent lifting with $\epsilon_1 = 1$ and $\epsilon_2 = 0$. Then*

- (a) *Let $\lambda \in N$. Then He_λ is the projective cover of a one-dimensional H -module.*

- (b) Let $\mu \in \widehat{G} \setminus N$. Then He_μ is the projective cover of a simple n_1 -dimensional H -module.
- (c) The isotypic component of He_μ is given by equivalence modulo X_1 , consisting of the projectives $He_\lambda, \lambda \in \mu X_1$.
- (d) The $e_\lambda, \lambda \in \widehat{G}$ are a complete set of primitive orthogonal idempotents in H .
- (e) Let $\lambda \in N$. Then $He_{\lambda X}$ is a block. The Gabriel quiver of this block has vertices corresponding to elements of λX and a pair of arrows

$$\mu \rightarrow \mu\chi_1$$

$$\mu \rightarrow \mu\chi_2$$

for every $\mu \in \lambda X$ (with doubled arrows if $\chi_1 = \chi_2$). The block $He_{\lambda X}$ is of wild representation type, unless $n_1 = n_2 = 2$, in which case it is of tame representation type.

- (f) Let $\mu \in \widehat{G} \setminus N$. Then $He_{\mu X}$ is a block. The Gabriel quiver of this block is cyclic and has vertices corresponding to cosets in $\mu X/X_1$ and arrows

$$\mu X_1 \rightarrow \chi_2 \mu X_1$$

corresponding to multiplication by χ_2 . The block $He_{\mu X}$ is a Nakayama algebra.

- (g) The $e_{\lambda X}$ indexed by the cosets $\lambda X \in \widehat{G}/X$ are a complete set of block idempotents of H .

PROOF: As noted above the subgroup N contains X , so that the equivalence classes $\lambda X, \lambda \in N$ form a partition of N ; taking $\lambda \in \widehat{G} \setminus N$ instead, we similarly see that the classes mod X form a partition of $\widehat{G} \setminus N$. By Lemma 2.2 the idempotents $e_{\lambda X}$ are all central. The two partitions combine to form a partition of \widehat{G} and we let

$$H_N = \sum_{\lambda \in N} He_\lambda$$

$$H_{\widehat{G} \setminus N} = \sum_{\lambda \in \widehat{G} \setminus N} He_\lambda$$

denote the complementary two-sided summands of H . Further let $J = xH_N + yH_N + yH_{\widehat{G} \setminus N}$, which is a two-sided ideal of H since H_N and $H_{\widehat{G} \setminus N}$ can be written as the sum of class subalgebras, each of the form $He_{\lambda X}$.

First suppose that $\lambda \in N$. Then $x^{n_1}e_\lambda = 0$ as $\lambda(a^{n_1}) = 1$. We also have $y^{n_2} = 0$, hence the image J in H_N is nilpotent. Thus J lies in the Jacobson radical of H_N . Next consider the projective modules He_λ . It is plain that $Je_\lambda = \oplus_{i+j>0} \mathbb{k}x^i y^j e_\lambda$, so He_λ/Je_λ is one-dimensional, spanned by a vector of weight λ . This finishes the proof of (a).

Secondly, suppose that $\mu \in \widehat{G}/N$ and consider the factor He_μ/Je_μ . Since $y^{n_2} = 0$, it is clear that the image of J in $H_{\widehat{G} \setminus N}$ is nilpotent. In this case

$x^{n_1}e_\mu$ is a nonzero scalar and

$$He_\mu/Je_\mu = \oplus_{0 \leq i \leq n_1-1} \mathbb{k}x^i e_\mu + Je_\mu$$

Observe that the action of x cyclically permutes the basis $\{x^i e_\mu + Je_\mu\}$, the vector $x^i e_\mu$ has weight $\mu\chi^i$ for all $i = 0, \dots, n_1 - 1$ and that these weights are pairwise distinct. It now follows easily that He_μ/Je_μ is a simple H -module. For future reference let us denote this simple module by L_μ . This finishes the proof of (b).

We show next that the isotypic component of L_μ consists of the simple modules $\{L_{\mu\chi} | \chi \in X_1\}$. Since the multiplicity of L_μ equals $\dim L_\mu = n_1$, we need to find n_1 modules isomorphic to L_μ . If $\mu = \mu'\chi_1^j$, then the H -module map specified by $e_\mu + Je_\mu \mapsto x^j e_{\mu'} + Je_{\mu'}$ gives an isomorphism $L_\mu \rightarrow L_{\mu'}$. This accounts for the isotypic component consisting of the n_1 mutually isomorphic projective (or simple) modules, which demonstrates (c).

To prove (d), first observe that the $e_\lambda, \lambda \in \widehat{G}$ are a complete set of orthogonal idempotents in $\mathbb{k}G$ by construction (§1.2). The proofs of parts (a) and (b) show the projective modules of He_λ are all simple modulo the nilpotent ideal J . Thus J is the radical of H , so the e_λ are all primitive as well.

The proof of (e) is similar to the nilpotent case addressed in the previous theorem, and will be omitted.

We prove (f). Observe that

$$Je_\mu/J^2e_\mu = \oplus_{0 \leq i < n_1} \mathbb{k}x^i y e_\mu + J^2e_\mu.$$

This factor is spanned by vectors $x^i y e_\mu + J^2e_\mu$ with weights $\mu\chi_2\chi_1^i$, which are permuted by the action of x , and annihilated by y . Thus it is apparent that Je_μ/J^2e_μ is isomorphic to $L_{\mu\chi_2}$. This shows that the Gabriel quiver is as asserted in (f) and also that the block containing L_μ consists of the simples indexed by $\mu X_1 X_2 = \mu X$. Similar observations show that $J^i e_\mu / J^{i+1} e_\mu$ is isomorphic to $L_{\mu\chi_2^i}$ for $i = 0, \dots, n_2 - 1$. Therefore He_μ is uniserial for all $\mu \in \widehat{G}/N$ and the proof of (f) is complete.

Finally, summing over the idempotents occurring in each block in the cases (e) and (f) (including multiplicities) produces every block idempotent as asserted in (g). This completes the proof of the theorem. \square

3.3. Unipotent unlinked liftings. We now pass to the remaining unipotent case with $\epsilon_i = 1$ for all i . In this case we require $\lambda(a^{n_1}) \neq 1 \neq \lambda(b^{n_2})$. It is easy to see that such λ exist. Recall $m = |X|$ and note that m divides $n_1 n_2$. Let $r = n_1 n_2 / m$.

Theorem 3.3. *Let H be an unlinked lifting and suppose $\epsilon_1 = 1 = \epsilon_2$ with $\lambda(a^{n_1}) \neq 1 \neq \lambda(b^{n_2})$. Then $He_{\lambda X}$ is a semisimple algebra isomorphic to the direct sum of r copies of $M_m(\mathbb{k})$.*

PROOF: Replace y by yb^{-1} so by abuse of notation we have $xy = yx$. We also abuse notation and write the images of generators in $He_{\lambda X}$ as the generators x, y themselves.

Since x^{n_1} and y^{n_2} are nonzero scalars, we can rescale and assume $x^{n_1} = y^{n_2} = 1$. Let U denote the subgroup of $He_{\lambda X}$ generated by x and y . For each $\alpha \in \widehat{U}$, let f_α be the primitive idempotent of $\mathbb{k}U$ associated to α . By Proposition 2.3 $He_{\lambda X}$ has basis

$$\{e_{\lambda\chi}x^{j_1}y^{j_2} | 0 \leq j_i \leq n_i, \chi \in X\},$$

so it follows that $He_{\lambda}f_\alpha$ has basis

$$\{e_{\lambda\chi}f_\alpha | \chi \in X\}$$

We claim that $He_{\lambda X}f_\alpha$ is a minimal left ideal. Let $\mu \in \lambda X$, and observe that the basis element $e_\mu f_\alpha$ has weight μ . Furthermore,

$$\begin{aligned} x^{j_1}y^{j_2}e_\mu f_\alpha &= e_{\mu\chi}x^{j_1}y^{j_2}f_\alpha \\ &= \alpha(x^{j_1}y^{j_2})e_{\mu\chi}f_\alpha. \end{aligned}$$

with $\chi = \chi_1^{j_1}\chi_2^{j_2}$. Therefore the given basis is permuted transitively by left multiplication by U and the basis elements have distinct G -weights. These facts imply that each $He_{\lambda X}f_\alpha$ is a minimal left ideal.

The decomposition

$$He_{\lambda X} = \bigoplus_{\alpha \in U} He_{\lambda X}f_\alpha$$

demonstrates the semisimplicity of $He_{\lambda X}$. Since $\dim He_{\lambda X} = n_1n_2m$ and $\dim He_{\lambda X}f_\alpha = m$, the proof is complete. \square

We can describe isotypic components for the unlinked unipotent lifting next. Let U be the subgroup generated by x, y as in the previous result, and let W be the kernel of the map

$$\pi : U \rightarrow X$$

given by $x \mapsto \chi_1, y \mapsto \chi_2$. For $\alpha \in \widehat{U}$, let $\alpha_W \in \widehat{W}$ denote the restriction of α to W .

Theorem 3.4. *Let $\alpha, \beta \in \widehat{W}$. Then $He_{\lambda X}f_\alpha \cong He_{\lambda X}f_\beta$ if and only if $\alpha_W = \beta_W$.*

PROOF: To prove this result it suffices to show that $f_\beta He_{\lambda X}f_\alpha = 0$ if and only if $\alpha_W \neq \beta_W$. To this end we calculate

$$\begin{aligned} f_\beta e_\mu f_\alpha &= \frac{1}{n_1n_2} \sum_{z \in U} \beta^{-1}(z) z e_\mu f_\alpha \\ (3.1) \quad &= \frac{1}{n_1n_2} \sum_{z \in U} (\beta^{-1}\alpha)(z) e_{\mu\pi(z)} f_\alpha \end{aligned}$$

for all $\mu \in \lambda X$, where the second expression is obtained by noting that $z e_\mu f_\alpha = e_{\mu\pi(z)} z f_\alpha = \alpha(z) e_{\mu\pi(z)} f_\alpha$.

Put $\omega = \beta^{-1}\alpha$ and assume that $\alpha_W \neq \epsilon$. Let T denote a transversal for W in U . Now we can write (3.1) in the form

$$\frac{1}{n_1 n_2} \sum_{t \in T} \omega(t) \left(\sum_{w \in W} \omega(w) \right) e_{\mu t} f_\alpha$$

which is zero by (1.4).

Moving to the opposite inclusion, suppose that $\omega_W = \epsilon$. Now (3.1) can be written

$$(3.2) \quad \frac{W}{n_1 n_2} \sum_{t \in T} \omega(t) e_{\mu \pi(t)} f_\alpha = \frac{|W|}{n_1 n_2 |G|} \sum_{g \in G} \mu(g^{-1}) \left[\sum_{t \in T} \omega(t) \pi(t)(g^{-1}) \right] g f_\alpha$$

using the definition of the e_λ . By duality for finite abelian groups $G/X^\perp \simeq \widehat{X}$ under the evaluation map $g \mapsto \text{ev}_g : \chi \mapsto \chi(g)$, $\chi \in X$. Since $U/W \simeq X$ there exists $g_0 \in G$ unique modulo X^\perp such that $\omega(t) = \pi(t)(g_0)$ for all $t \in T$. Therefore the inner sum in (3.2) becomes

$$(3.3) \quad \sum_{t \in T} \pi(t)(g_0 g^{-1})$$

As t runs over T , $\pi(t)$ runs over X . Thus the sum in (3.3) is in fact

$$\sum_{\phi \in X} \phi(g_0 g^{-1}) = \begin{cases} 0, & g_0 g^{-1} \notin X^\perp \\ |X|, & g_0 g^{-1} \in X^\perp \end{cases}$$

by using (1.4) and duality $G/X^\perp \simeq \widehat{X}$ mentioned above. Thus we can harmlessly assume that g runs over $g_0 X^\perp$ and then (3.2) reduces to

$$\frac{1}{|G|} \sum_{h \in X^\perp} \mu^{-1}(g_0 h) g_0 h f_\alpha = \frac{|X^\perp|}{|G|} g_0 \mu^{-1}(g_0) e_{\lambda X} f_\alpha,$$

the second equality by Lemma 2.4. This expression is clearly nonzero, so the proof is complete. \square

4. REPRESENTATIONS OF LINKED LIFTINGS

4.1. Linked liftings. Let H be a linked lifting with datum \mathcal{D} . Setting $q = \chi_2(a)$, $\chi = \chi_1 = \chi_2^{-1}$, and $n = |q|$ one can see readily that $\chi(a) = \chi(b) = q^{-1}$ and $n_1 = n_2$. Modifying the defining relations in 2.1 we arrive at a presentation of $H = H(\mathcal{D})$ by G together with the relations

$$(4.1) \quad x^n = \epsilon_1(a^n - 1)$$

$$(4.2) \quad y^n = \epsilon_2(b^n - 1)$$

$$(4.3) \quad gx = \chi(g)xg$$

$$(4.4) \quad gy = \chi^{-1}(g)y g$$

$$(4.5) \quad xy - qyx = \gamma(ab - 1)$$

with the coalgebra structure as given in 2.1.

A very important feature of linked liftings is the presence of a special central element analogous to the Casimir element of the classical Lie theory.

Lemma 4.1.

$$(4.6) \quad C = (a^{-1}x)y - \frac{\gamma}{q-1}(a^{-1} + qb)$$

$$(4.7) \quad = y(a^{-1}x) - \frac{\gamma}{q-1}(b + qa^{-1}).$$

is a central element of H

PROOF: The second formula follows from the first by (4.5). The first claim of the lemma is a straightforward verification. We sketch details. First, by (4.3) and (4.4) xy commutes with every $g \in G$. Further,

$$\begin{aligned} a^{-1}x(yx) &= xqa^{-1}(yx) \\ &= x(qa^{-1}(q^{-1}xy - q^{-1}\gamma(ab - 1))) \\ &= x(a^{-1}xy - \gamma(b - a^{-1})). \end{aligned}$$

by (4.3) and (4.5), and

$$(a^{-1} + qb)x = x(qa^{-1} + b).$$

by (4.3). It follows that

$$\begin{aligned} Cx &= x(a^{-1}xy - \gamma b(1 + \frac{1}{q-1}) - \gamma a^{-1}(\frac{q}{q-1} - 1)) \\ &= xC. \end{aligned}$$

The equality $yC = Cy$ is established similarly using the second expression for C . \square

We proceed to a very useful property of linked liftings.

Proposition 4.2. *Every linked lifting of a quantum plane is a symmetric algebra.*

PROOF: Let H be a linked lifting of the quantum plane. By a well-known result in [17] the assertion is equivalent to H being unimodular with inner square of the antipode.

We first show that S^2 is inner. It is easy to see that $S^2(x) = -a^{-1}xa$ and $S^2(y) = -b^{-1}yb$. Since $ab^{-1} \in \ker \chi$, ab^{-1} is central in H . Therefore $S^2(y) = a^{-1}ya$ as well. It follows immediately that $S^2(h) = -a^{-1}ha$ for all $h \in H$.

It remains to show that H is unimodular. To do this we provide an integral $I := ex^{n-1}y^{n-1}$ where $e = |G|^{-1} \sum_{g \in G} g$.

Let $g \in G$. Since $\chi_1\chi_2 = \epsilon$, it is immediate that $x^{n-1}y^{n-1}$ commutes with g . Hence

$$gI = Ig = \epsilon(g)I.$$

We show that $xI = Iy = 0$ next. This is clear if $x^n = 0 = y^n$, so assume, say, $x^n = a^n - 1$ with $\chi^n = \epsilon$. Then $\chi(a^n) = 1$ and hence a^n is central in H . Therefore

$$\begin{aligned} xI &= x(x^{n-1}y^{n-1}e) \\ &= x^n y^{n-1}e \\ &= (a^n - 1)y^{n-1}e \\ &= 0. \end{aligned}$$

Similarly $Iy = 0$.

It remains to show that $Ix = 0 = yI$. We shall prove $yI = 0$.

Note that as $x^{n-1}y^{n-1}$ has weight ϵ , $[y, x^{n-1}y^{n-1}]_c = yx^{n-1}y^{n-1} - x^{n-1}y^n$. Therefore

$$yI = x^{n-1}y^n e + [y, x^{n-1}y^{n-1}]_c e.$$

Since $y^n = \epsilon_2(b^n - 1)$ we have $y^n e = 0$. Furthermore, using Proposition 1.4 we have for the second term

$$\begin{aligned} [y, x^{n-1}y^{n-1}]_c &= [y, x^{n-1}]_c y^{n-1} + \chi^{n-1}(b)x^{n-1}[y, y^{n-1}]_c \\ &= \gamma q^{-(n-1)}(n-1)_q x^{n-2}(q^{-n+2}ab - 1)y^{n-1} \\ &\quad + \chi^{n-1}(b)x^{n-1}(1 - q^{-(n-1)})y^n \end{aligned}$$

Since $aby^{n-1} = q^{2(n-1)}y^{n-1}ab$ and $y^n e = 0$ we conclude that for some $h \in H$

$$yI = h(ab - 1)e$$

which is zero, completing the proof of the Theorem. \square

We proceed to a description of the class subalgebras of H . Let X be the subgroup of G generated by χ and put $N = |X|$. Since $q = \chi^{-1}(a)$, $n|N$. We set $m = N/n$.

Theorem 4.3. *Let H be a linked lifting and $\lambda \in \widehat{G}$.*

(a) *The algebra $He_{\lambda X}$ is generated by elements E, F, K subject to the relations*

$$(4.8) \quad E^n = \pm \epsilon_1 \lambda (a^{-n} - 1)$$

$$(4.9) \quad F^n = \epsilon_2 \lambda (b^n - 1)$$

$$(4.10) \quad K^N = \kappa$$

$$(4.11) \quad EF - FE = \eta(K^{-m} - K^m)$$

$$(4.12) \quad KE = \theta EK$$

$$(4.13) \quad KF = \theta^{-1}FK$$

where θ is a primitive N^{th} root of one, κ, η are nonzero scalars and $\epsilon_i = 0, 1$.

(b) *The algebra $He_{\lambda X}$ is of dimension Nn^2 .*

PROOF: We begin by modifying the defining relations of H . Let $x' = a^{-1}x$. Then (4.5) becomes

$$(4.14) \quad x'y - yx' = \gamma(b - a^{-1}).$$

A simple calculation gives $(ax')^n = q^{\binom{n}{2}} a^n x'^n = \pm a^n x'^n$. By rescaling x' if necessary we may assume that (4.1) holds for x' in place of x .

Let $E = e_{\lambda X} x'$ and $F = e_{\lambda X} y$. Multiplying (4.1) and (4.2) by $e_{\lambda X}$ and noting that $a^n, b^n \in X^\perp$, we obtain (4.8) and (4.9). Further, note that $\chi(G)$ is finite, hence cyclic subgroup of \mathbb{k}^\bullet and that $q \in \chi(G)$ has order n . So we can choose $\theta \in \mathbb{k}$ to be a generator of $\chi(G)$ and $g \in G$ such that

$$\begin{aligned} \chi(g) &= \theta \\ \theta^m &= q. \end{aligned}$$

Since $\chi(G) = G/X^\perp$, θ has order N . Also $a^{-1} = h_1 g^m$ and $b = h_2 g^{-m}$ for some $h_i \in X^\perp$.

Let $\bar{g} = e_{\lambda X} g$, $\alpha = \lambda(h_1)$ and $\beta = \lambda(h_2)$. Multiplying (4.14) by $e_{\lambda X}$ and using Lemma 2.4 we obtain

$$EF - FE = \gamma(\beta \bar{g}^{-m} - \alpha \bar{g}^m).$$

Pick $\mu \in \mathbb{k}$ such that $\mu = \alpha/\beta$ and set $K = \mu \bar{g}$. One can check that

$$\beta \bar{g}^{-m} - \alpha \bar{g}^m = \sqrt{\alpha\beta}(K^{-m} - K^m).$$

Setting $\eta = \gamma\sqrt{\alpha\beta}$ results in (4.11). This completes the proof of (a).

It is clear that any algebra so presented has dimension at most Nn^2 . In view of the fact that $[G : X^\perp] = |\hat{X}| = N$ and Proposition 2.3 we see that the dimension of $He_{\lambda X}$ is precisely this upper bound. \square

4.2. Casimir element. Write C_λ for the image $Ce_{\lambda X}$ of C in $He_{\lambda X}$. In this subsection we determine the minimal polynomial for each C_λ . First, we set up some notation. Explicitly we have

$$(4.15) \quad C_\lambda = EF - \frac{\eta}{q-1}(K^m + qK^{-m})$$

$$(4.16) \quad = FE - \frac{\eta}{q-1}(K^{-m} + qK^m).$$

Let $\eta' = \frac{\eta}{q-1}$ and $D = K^{-m} + qK^m$. Denote by A_λ the subalgebra of $He_{\lambda X}$ generated by K and define the \mathbb{k} -automorphism of A_λ by the rule $\tau(K^i) = \theta^i K^i$. Notice that the action of τ extends to an automorphism of $A_\lambda[t]$ with fixed ring $\mathbb{k}[t]$ and $aE = E\tau(a)$ for all $a \in A_\lambda$. Further, set

$$(4.17) \quad R_{\kappa, N} := \{\alpha \in \mathbb{k} \mid \alpha^N = \kappa\} \text{ and } R_N = R_{1, N}.$$

For every $\rho \in R_{\kappa, N}$, let $\phi_\rho : A_\lambda \rightarrow \mathbb{k}$ denote the evaluation homomorphism given by sending K to ρ .

Lemma 4.4. *For every $\rho \in R$*

$$f_\lambda(t) = \prod_{i=0}^{n-1} (t + \eta' \phi_{\theta^i \rho}(D)) - E^n F^n$$

is the minimal polynomial of C_λ over \mathbb{k} .

PROOF: We first note that the minimal polynomial of C_λ is of degree n . For otherwise, suppose $g(C_\lambda) = 0$ for some monic $g(t) \in \mathbb{k}[t]$ with $\deg g(t) = m < n$. By (4.11) and Lemma 4.11

$$(EF)^m = E^m F^m + \sum_{1 \leq i < m} a_i E^i F^i$$

for some $a_i \in A_\lambda$. Hence $C_\lambda^m = E^m F^m + \sum_{i < m} b_i E^i F^i$ for some $b_i \in A_\lambda$ and we deduce that

$$E^m F^m + \sum_{i < m} c_i E^i F^i = 0$$

for some $c_i \in A_\lambda$. But this contradicts the fact that $\{E^i F^j K^k | 0 \leq i, j, k < n\}$ is a basis for $He_{\lambda X}$.

We show next that $f_\lambda(C_\lambda) = 0$. Notice that from the definition of τ we have $aE^i = E^i \tau^i(a)$ for all $a \in A_\lambda$. Also, by the definition of C_λ , $FE = C_\lambda + \eta' D$. More generally for all $i > 0$,

$$\begin{aligned} F^i E^i &= F^{i-1} (FE) E^{i-1} \\ &= F^{i-1} E^{i-1} (C_\lambda + \eta' \tau^{i-1}(D)). \end{aligned}$$

Iterating we see that

$$F^n E^n = (C_\lambda + \eta' D)(C_\lambda + \eta' \tau(D)) \cdots (C_\lambda + \eta' \tau^{n-1}(D)).$$

This says precisely that C_λ satisfies $f_\lambda(t)$.

It remains to show that $f_\lambda(t) \in \mathbb{k}[t]$. Write $f_\lambda(t) = \sum_{j=0}^n e_j t^{n-j} - E^n F^n$ where e_j is the j th elementary symmetric function of $\eta' \tau^i(D)$. As τ has finite order, the $\eta' \tau^i(D)$ are permuted by τ , hence the e_j are fixed by it, and so $e_j \in \mathbb{k}$.

On the other hand a direct inspection using $\theta^m = q$ gives $\phi_\rho(\tau^i(D)) = \phi_{\theta^i \rho}(D)$. Since $f_\lambda(t) \in \mathbb{k}[t]$ and $E^n F^n \in \mathbb{k}$ we compute

$$\begin{aligned} f_\lambda &= \phi_\rho(f_\lambda(t)) = \prod_{i=0}^{n-1} (t + \eta' \phi_\rho(\tau^i(D))) - E^n F^n \\ &= \prod_{i=0}^{n-1} (t + \eta' \phi_{\theta^i \rho}(D)) - E^n F^n \end{aligned}$$

□

4.3. The unipotent case.

Theorem 4.5. *Suppose $E^n F^n \neq 0$. Then $He_{\lambda X}$ is isomorphic to the algebra of $n \times n$ matrices over $\mathbb{k}[C_\lambda]$.*

PROOF: Since E^n and F^n are both nonzero by hypothesis, we have $\epsilon_1 = \epsilon_2 = 1$ in the datum for H . It follows that $\chi^n = \epsilon$ and hence $N = n$. Let $B^- = \mathbb{k}[K, F]$ denote the algebra generated by K and F . It is easy to see that B^- is isomorphic to the skew group algebra where the group generated by K acts faithfully on $\mathbb{k}[F]$ by multiplication by powers of q^{-1} . By a standard argument e.g. [18], we obtain $B^- \cong M_n(\mathbb{k})$.

Since F is invertible, we have $E = F^{-1}C_\lambda + \eta' F^{-1}D$. Therefore $He_{\lambda X}$ is generated by K, F, C_λ . Since $\dim He_{\lambda X} = n^3$, we see that

$$\{K^i F^j C_\lambda^k | 0 \leq i, j, k < n\}$$

is a basis for $He_{\lambda X}$. It follows that $He_{\lambda X} \cong B^- \otimes \mathbb{k}[C_\lambda] \cong M_n(\mathbb{k}[C_\lambda])$. \square

Corollary 4.6. *Suppose $E^n F^n \neq 0$. Then every simple $He_{\lambda X}$ -module is n -dimensional.* \square

4.4. The semi-nilpotent case. We now consider the case $F^n \neq 0$ and $E^n = 0$ with varying assumptions on κ . As in the previous theorem we have $n = N$. In the following theorem we put

$$V = \mathbb{k}[v]/(v^2)$$

for the truncated polynomial algebra. For an algebra A we let A^j denote the direct sum of j copies of the A .

Theorem 4.7. *Suppose $F^n \neq 0$ and $E^n = 0$.*

(a) *If $\kappa^2 \neq 1$, then $He_{\lambda X}$ is isomorphic to $M_n(\mathbb{k})^n$.*

Assume now that $\kappa^2 = 1$

(b) *If n is odd, then $He_{\lambda X}$ is isomorphic to*

$$M_n(\mathbb{k}) \oplus M_n(V)^{\frac{n-1}{2}}.$$

Assume further that n is even

(c) *If $\kappa = 1$, then $He_{\lambda X}$ is isomorphic to*

$$M_n(V)^{\frac{n}{2}}.$$

(d) *If $\kappa = -1$, then $He_{\lambda X}$ is isomorphic to*

$$M_n(\mathbb{k})^2 \oplus M_n(V)^{\frac{n-2}{2}}.$$

PROOF: The argument used in the preceding Theorem shows that $He_{\lambda X} \cong M_n(\mathbb{k}[C_\lambda])$. Let $\rho \in R_{\kappa, n}$. Then $R_{\kappa, n} = \{q^i \rho | i = 0, 1, \dots, n-1\}$. As $E^n = 0$, we obtain the factorization

$$f_\lambda = \prod_{\rho \in R_{\kappa, n}} (t + \eta' \phi_\rho(D)) = \prod_{\rho \in R_{\kappa, n}} (t + \eta' D(\rho)),$$

using the fact that $\theta = q$. So the asserted results in (a)- (d) are a question of determining the multiplicity of the roots $-\eta'(D(\rho))$. A simple computation gives that for $\rho_1, \rho_2 \in R_{\kappa, n}$,

$$(4.18) \quad D(\rho_1) = D(\rho_2) \Leftrightarrow \rho_1 \rho_2 = q^{-1}.$$

In case $\kappa^2 \neq 1$ this condition does not hold for any two roots, whence we get (a).

Suppose $\kappa^2 = 1$. Then (4.18) can be rephrased by stating that the root $-\eta'D(\rho)$ occurs with multiplicity one if and only if $\rho^2 = q^{-1}$. Otherwise the root $-\eta'D(\rho) = -\eta'D(q^{-1}\rho^{-1})$ occurs twice. It follows that the primary decomposition of $\mathbb{k}[C_\lambda]$ is

$$(4.19) \quad \mathbb{k}[C_\lambda] = \mathbb{k}^s \oplus (\oplus \mathbb{k}[t]/(t + \eta'D(\rho))^2$$

where s is the number of simple roots.

It remains to determine s . We note that for $\rho, \zeta \in R_{\kappa, n}$ $\zeta \rho^{-1} \in R_n$ and $\rho^2 = \zeta^2$ if and only if $(\zeta \rho^{-1})^2 = 1$. The mapping $\sigma : R_n \rightarrow R_n$, $q^i \mapsto q^{2i}$ is an isomorphism if n is odd, while $\text{im } \sigma = R_n^2$ for even n . If $\kappa = -1$, let π be a primitive $2n^{\text{th}}$ root of 1 such that $\pi^2 = q$. Then the preceding paragraph shows that $s = 1$ if n is odd, $s = 0$ if n is even and $\kappa = 1$, and $s = 2$ if n is even and $\kappa = -1$. \square

4.5. Standard cyclic modules. We continue with the linked case in summands $He_{\lambda X}$ with $E^n = 0$. We explicitly construct some induced modules which turn out to be simple for the case where $F^n \neq 0$. In the next section we will assume $E^n = 0 = F^n$ and there these modules will play a crucial role. Let B^+ denote the subalgebra of $He_{\lambda X}$ generated by E, K . For every $\rho \in R_{\kappa, N}$ we make \mathbb{k} a B^+ - module denoted by \mathbb{k}_ρ by setting $K.1_\rho = \rho$ and $E.1_\rho = 0$ where 1_ρ is identified with $1 \in \mathbb{k}$. We define the $He_{\lambda X}$ - module $Z(\rho)$ by

$$Z(\rho) = He_{\lambda X} \otimes_{B^+} \mathbb{k}_\rho$$

Since $He_{\lambda X}$ is a free B^+ - module with basis $\{F^i | 0 \leq i \leq n-1\}$, we have a \mathbb{k} - basis for $Z(\rho)$

$$\{F^i \otimes 1_\rho\}$$

and we shall write $w_i = F^i \otimes 1_\rho$. This is the *standard basis* for $Z(\rho)$. Using a conventional argument, it follows that each simple module is a homomorphic image of some $Z(\rho)$. When F is invertible and the $F^i \otimes 1_\rho$ have distinct K -weights, it is immediate that $Z(\rho)$ is simple.

Proposition 4.8. *In case $E^n = 0$, $\{Z(\rho) | \rho \in R\}$ is a set of representatives of simple $He_{\lambda X}$ - modules. In addition, $Z(\rho) \cong Z(\zeta)$ if and only if $C_\lambda.1_\rho = C_\lambda.1_\zeta$.*

PROOF: For every $\rho \in R_{\kappa, n}$ we put C_ρ for the summand of 4.19 corresponding to the root $-\eta'D(\rho)$ and let ϵ_ρ be the primitive idempotent generating C_ρ . Clearly $\epsilon_\rho I_n$ is the unity of $M_n(C_\rho)$. Since $M_n(C_\rho)$ has a unique isomorphism class of simple modules, $Z(\rho)$ is in the class if and only if the restriction

$\epsilon_\rho|_{Z(\rho)}$ is the identity. Since ϵ_ρ is a polynomial in C_λ taking the value 1 at $-\eta'D(\rho)$ and zero at any other root of $f_\lambda(t)$, and as $C_\lambda \cdot 1_\rho = -\eta'D(\rho)$, the result follows. \square

If we assume that $F^n = 0$ instead we can construct a $He_{\lambda X}$ -module in the obvious manner, inducing from the subalgebra B^- and obtain $He_{\lambda X}$ -modules which we denote by $Z'(\rho)$. These modules are simple if $E^n \neq 0$. We have a \mathbb{k} -basis for $Z'(\rho)$

$$\{E^i \otimes 1_\rho | 0 \leq i \leq n-1\}$$

and we shall write $v_i = E^i \otimes 1_\rho$. This is called the *standard basis* for $Z'(\rho)$.

4.6. The nilpotent case. From now on we assume $E^n = 0 = F^n$. We use the notation of the previous subsection, including the modules $Z'(\rho)$ and $Z(\rho)$ and their standard bases. Recall that $N = nm$ and the definition of $R_{\kappa, N}$. We will refer to elements of $R_{\kappa, N}$ as roots.

Definition 4.9. Let $p : R_{\kappa, N} \rightarrow R_{\kappa^2, n}$ be defined by $p(\rho) = \rho^{2m}$. We call $\rho \in R_{\kappa, N}$ *exceptional* if $\rho^{2m} = q^{n-1}$. For every $\rho \in R_{\kappa, N}$ we define integers $e(\rho)$ and $e'(\rho)$ by the rule

$$p(\rho) = q^{e(\rho)} = q^{-e'(\rho)}.$$

Definition 4.10. Let M be an $He_{\lambda X}$ -module. We say that $0 \neq v \in M$ has *weight* $\pi \in R$ if $K.v = \pi v$. A weight vector v is said to be *E-trivial* if $E.v = 0$.

We preface a description of the induced modules with a commutation formula similar to one for quantized universal enveloping algebra for \mathfrak{sl}_2 [12]. The proof is entirely similar to that of Proposition 1.4 (b) and will be omitted.

Lemma 4.11. *For every $s \geq 1$*

- (1) $[E, F^s] = \eta(s)_q F^{s-1} (K^{-m} - q^{-(s-1)} K^m).$
- (2) $[F, E^s] = \eta(s)_q E^{s-1} (K^m - q^{-(s-1)} K^{-m}).$

Proposition 4.12. (a) $Z'(\rho)$ is simple if and only if $e'(\rho) = n-1$. If not, $Z'(\rho)$ has a unique nonzero proper submodule generated by $v_{e'+1}$.
 (b) $Z(\rho)$ is simple if and only if $e(\rho) = n-1$. If not, $Z(\rho)$ has a unique nonzero proper submodule generated by w_{e+1} .

PROOF: Recall the standard basis $v_i = E^i \otimes 1_\rho$, $i = 0, 1, \dots, n-1$. We show first that $Z'(\rho)$ has a nonzero proper submodule if and only if there is an F -trivial v_s with $s > 0$.

Since v_i has weight $\theta^i \rho$, the v_i have distinct weights. Thus, a proper submodule M of $Z'(\rho)$ is the span of the v_i that lie in M . Let $s = \min\{i | v_i \in M\}$. By Lemma 4.11 (2) $F.v_s = 0$, and $s > 0$ because M is proper. Conversely, if $F.v_s = 0$ for some $s > 0$, then $He_{\lambda X} v_s$ is a nonzero proper submodule of $Z'(\rho)$.

Using Lemma 4.11(2) we quickly deduce that (for $s > 0$) $F.v_s = 0$ if and only if $\rho^{2m} = q^{1-s}$. Setting $e' = s - 1$, we arrive at the desired equivalence.

It follows from $\rho^{2m} = q^{1-s}$ that $He_{\lambda X}v_s$ contains no F -trivial vectors other than v_s . Thus $He_{\lambda X}v_s$ is the unique proper nonzero submodule of $Z'(\rho)$.

The proof of (b) is similar using Lemma 4.11 (1). \square

We dispose easily of the case where $\kappa^2 \neq 1$.

Theorem 4.13. *Assume that $\kappa^2 \neq 1$. then $He_{\lambda X} \cong M_n(\mathbb{k})^N$.*

PROOF: Suppose that $Z(\rho)$ is not simple. Then the result above says that $\rho^{2m} = q^e$, which implies $\kappa^2 = \rho^{2N} = \rho^{2mn} = q^{en} = 1$. We conclude that every $Z(\rho)$ is simple. Furthermore, since each $Z(\rho)$ contains a unique E -trivial vector of weight ρ , we see that the $Z(\rho)$ are pairwise nonisomorphic. As $\dim He_{\lambda X} = Nn^2$, the proof is complete. \square

4.7. $\kappa^2 = 1$. Let $\text{rad } Z(\rho)$ (resp. $\text{rad } Z'(\rho)$) denote the proper (possibly zero) submodule of $Z(\rho)$ (resp. $Z'(\rho)$). Further let

$$L(\rho) = Z(\rho)/\text{rad } Z(\rho).$$

$$L'(\rho) = Z'(\rho)/\text{rad } Z'(\rho).$$

Proposition 4.14. *Assume $\kappa^2 = 1$.*

- (a) *The simple modules $L(\rho), \rho \in R_{\kappa, N}$ are a full set of representatives of simple $He_{\lambda X}$ -modules.*
- (b) *The simple modules $L'(\rho), \rho \in R_{\kappa, N}$ are another set of representatives of simple $He_{\lambda X}$ -modules.*
- (c) *If $Z(\rho)$ is not simple, $Z(\rho)$ is a nonsplit extension of $L(\theta^{-(e(\rho)+1)}\rho)$ by $L(\rho)$.*
- (d) *If $Z'(\rho)$ is not simple, $Z'(\rho)$ is a nonsplit extension of $L(\theta^{n-1}\rho)$ by $L(\theta^{e'(\rho)}\rho)$.*
- (e) $\dim L(\rho) = e(\rho) + 1$.
- (f) $\dim L'(\rho) = e'(\rho) + 1$.

PROOF: (a)-(b). Every simple $He_{\lambda X}$ -module contains an E -trivial vector v , say of weight $\rho \in R_{\kappa, N}$. The map given by $1_\rho \mapsto v$ induces a nontrivial homomorphism $Z(\rho) \rightarrow M$. Since $L(\rho)$ contains a unique line of E -trivial vectors of weight ρ , $L(\rho) \not\cong L(\rho')$ for $\rho \neq \rho'$. This proves (a). The proof of (b) is similar.

(c) and (e). By Proposition 4.12 (b) $\text{rad } Z(\rho)$ is generated by an E -primitive $w_{e(\rho)+1}$, hence is the span of $\{w_i | e(\rho) + 1 \leq i \leq n-1\}$. This proves (e). Further, the weight of $w_{e(\rho)+1}$ is $\theta^{-(e(\rho)+1)}\rho$, so that $\text{rad } Z(\rho) \cong L(\theta^{-(e(\rho)+1)}\rho)$. This proves (c).

(d) and (f). By Proposition 4.12 (a) $\text{rad } Z'(\rho)$ is the span of $\{v_i | e'(\rho) + 1 \leq i \leq n-1\}$. Now, up to a scalar multiple, the only F -trivial weight vector of $\text{rad } Z'(\rho)$ is v_{n-1} and the only F -trivial weight vector in $L'(\rho)$ is

$v_{e'} + \text{rad } Z'(\rho)$. These vectors have weights $\theta^{e'(\rho)}\rho$ and $\theta^{n-1}\rho$, respectively. This establishes (d) and (f). \square

We define a mapping $\sigma : R_{\kappa,N} \rightarrow R_{\kappa,N}$ by

$$\sigma(\rho) = \begin{cases} \theta^{-(e(\rho)+1)}\rho & \text{if } \rho \text{ is nonexceptional} \\ \sigma(\rho) = \rho & \text{otherwise.} \end{cases}$$

Lemma 4.15. (a) σ is a permutation of order $2m$.

(b) The orbits of $R_{\kappa,N}$ under σ are of size $2m$ or 1 .

(c) $\sigma^{-1}(\rho) = \theta^{n-e(\rho)-1}\rho$.

PROOF: (a)-(b). Suppose ρ is nonexceptional. By the two previous propositions $e(\sigma(\rho)) + 1 + e(\rho) + 1 = n$. This equality gives

$\sigma^2(\rho) = \theta^{-(e(\sigma(\rho))+1)}\sigma(\rho) = \theta^{-n}\rho$. We conclude that σ is a bijection and $\sigma^{2s}(\rho) = \theta^{-sn}\rho$ for every integer $s \geq 0$ hence $\sigma^{2m} = 1$. Suppose that for some odd number $1 + 2s < 2m$ $\sigma^{1+2s}(\rho) = \rho$. Then

$$\rho = \sigma(\sigma^{2s}(\rho)) = \sigma(\theta^{-sn}\rho) = \theta^{-(e(\rho)+1)}\theta^{-sn}\rho$$

because $e(\theta^{-sn}\rho) = e(\rho)$. This implies $e(\rho)+1 \equiv 0 \pmod n$ which contradicts the fact that $e(\rho) \leq n-2$.

(c) To prove (c), notice that $\sigma^{-2}(\rho) = \theta^n\rho$. So $\sigma^{-1}(\rho) = \sigma(\theta^n\rho) = \theta^{n-e(\rho)-1}\rho$ as asserted. \square

Proposition 4.16. For every nonexceptional root ρ , $Z'(\theta^{-e(\rho)}\rho)$ is a non-split extension of $L(\sigma^{-1}(\rho))$ by $L(\rho)$. When $\sigma = \sigma^{-1}$, $Z'(\theta^{-e(\rho)}\rho) \not\cong Z(\rho)$.

PROOF: Set $\zeta = \theta^{-e(\rho)}\rho$ and consider $Z'(\zeta)$. By Proposition 4.14 $Z'(\zeta)$ is a nonsplit extension of $L(\theta^{n-1}\zeta)$ by $L(\theta^{e'(\zeta)}\zeta)$. Now $\zeta^{2m} = q^{-2e(\rho)}q^{e(\rho)} = q^{-e(\rho)}$ so by the definition of $e'(\zeta)$ we have $e'(\zeta) = e(\rho)$. Therefore $\theta^{e'(\zeta)}\zeta = \rho$ while $\theta^{n-1}\zeta = \sigma^{-1}(\rho)$ by part (c) of the preceding Lemma..

In the case where $\sigma = \sigma^{-1}$ (i.e. $m = 1$) we have two modules with the same composition series as in the preceding result, namely $Z'(\theta^{-e(\rho)}\rho)$ and $Z'(\rho)$. These modules are nonisomorphic. For, by definition, $Z(\rho)$ has an E -trivial generator, while $Z'(\theta^{-e(\rho)}\rho)$ does not. \square

We need one more result before the main theorem. This is

Lemma 4.17. (1) If n is odd, then for every d , $1 \leq d \leq n$ there are exactly m nonisomorphic simple modules of dimension d .

(2) If n is even and $\kappa = 1$, then for every odd d , $1 \leq d \leq n$ there are exactly $2m$ nonisomorphic simple modules of dimension d .

(3) If n is even and $\kappa = -1$, then for every even d , $1 \leq d \leq n$ there are exactly $2m$ nonisomorphic simple modules of dimension d .

PROOF: By Proposition 4.14 $\dim L(\rho) = d$ if and only if $e(\rho) = d-1$. So the asserted results are a question of the number of solutions in $R_{\kappa,N}$ of the equation $\rho^{2m} = q^e$. Recall the mapping $p : R_{\kappa,N} \rightarrow R_n$, $p(\rho) = \rho^{2m}$. For $\rho, \zeta \in R_{\kappa,N}$ $p(\rho) = p(\zeta)$ if and only if $p(\rho\zeta^{-1}) = 1$. Now $\rho\zeta^{-1} \in R_N$,

and as R_N is generated by θ with $\theta^m = q$, the image of p restricted to R_N is the subgroup of R_n generated by q^2 . This proves parts (1) and (2) as $p(R_{\kappa,N}) = p(R_N)$ in both cases.

Assume conditions of part (3). Pick an element $\pi \in \mathbb{k}$ such that $\pi^2 = \theta$. Clearly $\pi \in R_{-1,N}$ and moreover every $\rho \in R_{-1,N}$ is of the form $\theta^r \pi$ for some $0 \leq r < N$. Now $p(\theta^r \pi) = q^{2r+1}$ which shows that $e(\rho)$ runs over all odd integers between 1 and n . This proves (3). \square

We let $P(\rho)$ denote the projective cover of $L(\rho)$. The i th term of the radical series of $P(\rho)$ shall be denoted by $\text{rad}^i P(\rho)$, and the socle of $P(\rho)$ shall be denoted by $\text{soc } P(\rho)$.

Theorem 4.18. (1) *For every nonexceptional root ρ*

- (a) $\dim P(\rho) = 2n$.
- (b) *The radical series of $P(\rho)$ is*

$$P(\rho) \supset \text{rad } P(\rho) \supset \text{soc } P(\rho) \supset 0$$

$$\text{with } \text{rad } P(\rho) / \text{rad}^2 P(\rho) \cong L(\sigma(\rho)) \oplus L(\sigma^{-1}(\rho)).$$

- (2) *If ρ is exceptional, then $P(\rho) = L(\rho)$.*

PROOF: Since H is symmetric algebra by Theorem 4.2 $\text{soc } P(\rho) = L(\rho)$. Let ρ be a nonexceptional root. By Propositions 4.14 and 4.16 there are epimorphisms of $P(\rho)$ on $Z(\rho)$, $Z'(\theta^{-e(\rho)}\rho)$ carrying $\text{rad } P(\rho)$ to $L(\sigma(\rho))$ and $L(\sigma^{-1}(\rho))$, respectively. Thus $\text{rad } P(\rho) / \text{rad}^2 P(\rho)$ has a summand $L(\sigma(\rho)) \oplus L(\sigma^{-1}(\rho))$. Therefore

$$\dim P(\rho) \geq 2 \dim L(\rho) + \dim L(\sigma^{-1}(\rho)) + \dim L(\sigma(\rho)) = 2n$$

where the last equality holds since $Z(\rho) / L(\sigma(\rho)) = L(\rho)$ and $Z'(\theta^{-e(\rho)}\rho) / L(\sigma^{-1}(\rho)) = L(\rho)$.

By general principles

$$(4.20) \quad He_{\lambda X} = \bigoplus_{\rho \in R_{\kappa,N}} \dim P(\rho)^{\dim L(\rho)}$$

If n is odd, Lemma 4.17 (1) gives

$$\begin{aligned} \sum_{\rho \in R_{\kappa,N}} \dim P(\rho) \cdot \dim L(\rho) &= \sum_{\rho \text{ nonexceptional}} \dim P(\rho) \cdot \dim L(\rho) \\ &\quad + \sum_{\rho \text{ exceptional}} \dim P(\rho) \cdot \dim L(\rho) \\ &\geq 2n(1 + 2 + \dots + n - 1)m + n^2 m = Nn^2. \end{aligned}$$

Since $\dim He_{\lambda X} = Nn^2$ that expression forces $\dim P(\rho) = 2n$ for all nonexceptional roots ρ and $\dim P(\rho) = n^2$ for the exceptional roots. This completes the proof. The even cases are treated similarly. \square

We denote by $B(\rho)$ the block of $He_{\lambda X}$ containing $P(\rho)$. We let \mathcal{W} denote group generated by σ .

Corollary 4.19. *The blocks of $He_{\lambda X}$ correspond to \mathcal{W} -orbits in $R_{\kappa, N}$. The correspondence is given by the formula*

$$B(\rho) = \sum_{\zeta \in \mathcal{W}\rho} P(\zeta)^{\dim L(\zeta)}$$

PROOF: Recall that two projective indecomposable modules P and P' are *linked* if they share a composition factor. Linkage gives rise to an equivalence relation on the set of projective indecomposable modules. The block of P is the sum of all projective indecomposable modules equivalent to P . By the preceding Theorem $P(\rho)$ is linked to $P(\zeta)$ if and only if $\zeta = \sigma^\pm(\rho)$. The formula follows. \square

Theorem 4.20. *Assume ρ is a nonexceptional root. Then*

- (a) *The Gabriel quiver Q_ρ of the block containing $P(\rho)$ is the quiver with vertices being the isomorphism classes of simples $[L(\sigma^i(\rho))]$, $i \in \mathbb{Z}/2m\mathbb{Z}$ with arrows*

$$[L(\sigma^i(\rho))] \begin{matrix} \xrightarrow{a_i} \\ \xleftarrow{b_i} \end{matrix} [L(\sigma^{i+1}(\rho))]$$

corresponding to translation by σ .

- (b) *The basic algebra of the block containing $P(\rho)$ is the quotient of the path algebra of Q_ρ with relations*

$$a_i b_i - b_{i+1} a_{i+1}$$

and all other paths of length ≥ 2 .

- (c) *The block of Q_ρ is a special biserial algebra and therefore of tame representation type.*

PROOF: For generalities on basic algebras and their presentations by quivers with relations, the reader may consult [5]. The present situation is similar to [11]; however, the argument there is not compatible with the theory developed here. We proceed as follows.

As a first case assume $m > 1$. By Theorem 4.18 $\text{rad } P(\rho) / \text{rad}^2 P(\rho)$ equals $L(\sigma(\rho)) \oplus L(\sigma^{-1}(\rho))$, therefore the quiver has arrows as in the statement. Let $\mathbb{k}Q_\rho$ be the path algebra of the quiver and \mathcal{B} be the basic algebra of $B(\rho)$. We want to construct an epimorphism of $\mathbb{k}Q_\rho$ onto \mathcal{B} (cf. [5, III 1.9]) whose kernel is generated by relations in (b). Thanks to Propositions 4.14 and 4.16 there are nonsplit extensions E_i and E'_i of $L(\sigma^i(\rho))$ by $L(\sigma^{i-1}(\rho))$ and $L(\sigma^i(\rho))$ by $L(\sigma^{i+1}(\rho))$, respectively. Since $P(\sigma^i(\rho))$ is injective both embed in $P(\sigma^i(\rho))$. Now we assign to a_i an epimorphism $P(\sigma^i(\rho)) \rightarrow E_{i+1}$ and to b_i an epimorphism $P(\sigma^{i+1}(\rho)) \rightarrow E'_i$ which we still denote by a_i and b_i . Since $\sigma^{i-1}(\rho) \neq \sigma^i(\rho), \sigma^{i+1}(\rho)$ we have $a_i(E_i) = 0$, hence $a_i a_{i-1} = 0$, and likewise $b_{i-1} b_i = 0$. Further $a_i b_i$ and $b_{i+1} a_{i+1}$ are endomorphisms of $P(\sigma^{i+1}(\rho))$ into its socle $L(\sigma^{i+1}(\rho))$. Therefore

$$\gamma_i a_i b_i = b_{i+1} a_{i+1}$$

for some $\gamma_i \in \mathbb{k}$ and all $i \in \mathbb{Z}/2m\mathbb{Z}$.

We wish to change the basis to get rid of the coefficients. We will follow argumentation of [11]. Replacing a_i by $a'_i := \gamma_i \cdots \gamma_{2m-1} a_i$, $i = 1, 2, \dots, 2m-1$, we obtain the relations

$$\begin{aligned} a'_i b_i &= b_{i+1} a'_{i+1} \text{ for } i = 1, \dots, 2m-2 \\ a'_{2m-1} b_{2m-1} &= b_0 a_0 \\ \gamma_0 \cdots \gamma_{2m-1} a_0 b_0 &= b_1 a'_1 \end{aligned}$$

We know by Theorem 4.2 that H is a symmetric algebra and then so are the blocks of $He_{\lambda X}$ and their basic algebras. Let $\psi : \mathcal{B} \rightarrow \mathbb{k}$ be a symmetrizing linear form. Then

$$\begin{aligned} \psi(\gamma_0 \cdots \gamma_{2m-1} a_0 b_0) &= \psi(b_1 a'_1) = \psi(a'_1 b_1) \\ &= \cdots = \psi(a'_{2m-1} b_{2m-1}) = \psi(b_0 a_0) = \psi(a_0 b_0) \end{aligned}$$

whence $(\gamma_0 \cdots \gamma_{2m-1} - 1)a_0 b_0 \in \ker \psi$ and this element spans a one-dimensional left ideal of H . Thus $\gamma_0 \cdots \gamma_{2m-1} = 1$ as desired.

Moving to the case $m = 1$ we let $i \in \mathbb{Z}/2\mathbb{Z}$. Now we have just two projective indecomposable modules in $B(\rho)$, call them P_0 and P_1 , and let $L_i = P_i / \text{rad } P_{i+1}$. In contrast to the previous case, there doesn't exist a uniform choice of length two indecomposable submodules in $\text{rad } P_i$. We argue as follows. As above by Proposition 4.16 there are nonisomorphic submodules E_1 and E'_1 in $\text{rad } P_1$ with the top composition factor L_0 . Consequently, there are epimorphisms $a_0 : P_0 \rightarrow E_1$ and $b_1 : P_0 \rightarrow E'_1$. Put $F_0 = \ker a_0$ and $F'_0 = \ker b_1$. Both F_0 and F'_0 lie in $\text{rad } P_0$ hence by Theorem 4.18 they are extensions of L_0 by L_1 . Therefore there are epimorphisms $a_1 : P_1 \rightarrow F_0$ and $b_0 : P_1 \rightarrow F'_0$; hence $a_0 a_1 = 0 = b_1 b_0$ holds by construction. However, since a_i, b_i are nonisomorphisms they send the socle of P_j to zero for all i . Therefore every path of length ≥ 2 is either zero or spans a 1-dimensional ideal of \mathcal{B} . Since \mathcal{B} is symmetric $a_1 a_0 = 0 = b_0 b_1$ as well, whence $E_1 \subset \ker a_1$ and $E'_1 \subset \ker b_0$. As $\dim \ker a_1 = n = \dim E_1$ we see that $E_1 = \ker a_1$ and similarly $E'_1 = \ker b_0$. From this we conclude that $F_0 \neq F'_0$. Indeed, since $\dim \text{Hom}(P_1, F_0) = [F_0 : L_1] = 1$ were $F_0 = F'_0$ we would have $a_1 = \alpha b_0, \alpha \in \mathbb{k}^\bullet$, which forces $E_1 = E'_1$, a contradiction. It follows that $a_0 b_0$ and $b_1 a_1$ are nonzero homomorphisms $P_1 \rightarrow \text{soc } P_1$, and similar conclusions hold for $b_0 a_0$ and $a_1 b_1$. Therefore there are $\gamma_i \in \mathbb{k}^\bullet$ such that

$$\begin{aligned} \gamma_1 a_0 b_0 &= b_1 a_1 \\ \gamma_2 b_0 a_0 &= a_1 b_1 \end{aligned}$$

But then the calculation

$$\begin{aligned} \psi(b_1 a_1) &= \gamma_1 \psi(a_0 b_0) = \psi(a_1 b_1) \\ &= \gamma_2 \psi(b_0 a_0) = \gamma_2 \psi(a_0 b_0) \end{aligned}$$

gives $\gamma_1 = \gamma_2$. Replacing a_0 by $\gamma_1 a_0$ we obtain the desired result. \square

The proof of (c) follows directly from the definition and theory of special biserial algebras, see [10]. \square

Lastly, we want to describe embeddings of blocks and projective indecomposable modules in H . First, we take the subalgebra A_λ of $He_{\lambda X}$ of section 4.2. From its definition we have $A_\lambda \cong \mathbb{k}[t]/(t^N - \kappa)$, hence $A_\lambda = \oplus \mathbb{k}e_\zeta$ where e_ζ is a primitive idempotent uniquely determined by the property $Ke_\zeta = \zeta e_\zeta$. Secondly, we use primitive idempotents ϵ_μ associated to roots $\mu \in R_{\kappa,n}$ defined in section 4.5. For every $\rho \in R_{\kappa,N}$, $\rho^m \in R_{\kappa,n}$ hence gives rise to the idempotent ϵ_{ρ^m} . Abusing notation we write the latter as ϵ_ρ .

Proposition 4.21. *Suppose ρ is a nonexceptional root. Then*

- (1) $B(\rho) = He_{\lambda X}\epsilon_\rho$.
- (2) *Every nonsimple projective indecomposable module of H is isomorphic to $He_{\lambda X}\epsilon_\rho e_\zeta$ for some λ, ρ, ζ .*

PROOF: It is elementary to see that for a root μ of the minimal polynomial $f_\lambda(t)$, $\epsilon_\rho(\mu) = 1$ if $\mu = -\eta'D(\rho)$, and zero, otherwise. For every simple $He_{\lambda X}$ -module $L(\zeta)$ C_λ acts on $L(\zeta)$ via multiplication by $-\eta'D(\zeta)$. Therefore ϵ_ρ acts on $L(\zeta)$ via multiplication by $\epsilon_\rho(-\eta'D(\zeta))$. Now $D(\zeta) = D(\rho)$ if and only if $\zeta^m = \rho^m$ or $\zeta^m \rho^m = q^{-1}$. Since ρ is nonexceptional, $q^{-1}\rho^{-m} \neq \rho^m$; hence both cases occur. Next observe that for every fixed element $\mu \in R_{\kappa,n}$ the set of solutions to the equation $x^m = \mu$ in $R_{\kappa,N}$ has m elements. Therefore $Y = \{\zeta | D(\zeta) = D(\rho)\}$ has $2m$ elements.

On the other hand by Lemma 4.15, $\zeta \in \mathcal{W} \cdot \rho$ if and only if $\zeta = \theta^{-np}\rho$ or $\zeta = \theta^{-np-e(\rho)-1}\rho$. In the first case $\zeta^m = \rho^m$ and in the second $\zeta^m \rho^m = q^{-1}$. Thus $\mathcal{W} \cdot \rho \subset Y$, hence by Lemma 4.15 (b) they are equal. It follows that ϵ_ρ acts as the identity on $B(\rho)$ and annihilates every other block, whence (1).

The number of projective indecomposable modules in a decomposition of $B(\rho)$ equals $\sum_{\zeta \in \mathcal{W} \cdot \rho} \dim L(\zeta)$. For every $\zeta \in \mathcal{W} \cdot \rho$, $\sigma(\zeta) \neq \zeta$, hence there are m distinct pairs $\{\zeta, \sigma(\zeta)\}$ in $\mathcal{W} \cdot \rho$. For each such pair Proposition 4.14 (c) gives $\dim L(\zeta) + \dim L(\sigma(\zeta)) = n$. Thus $B(\rho)$ is the direct sum N indecomposable summands. Since we have N idempotents e_ζ , the result follows. \square

We turn to the case of a projective simple module $P(\rho)$.

Proposition 4.22. *Suppose ρ is an exceptional root. Let $f = e_\rho E^{n-1} F^{n-1}$. Then*

$$He_{\lambda X}f \cong P(\rho)$$

and $He_{\lambda X}f$ is a direct summand of $He_{\lambda X}$.

PROOF: Evidently f has weight ρ . Further, it is elementary to derive the formula $e_\rho = \frac{\rho}{N\kappa} \prod_{\zeta \neq \rho} (K - \zeta)$ and then a simple calculation gives $Ee_\rho = e_{\theta\rho}E$, hence f is E -trivial, which proves the first assertion.

Next we adjust Kac's formula [8, (1.3.1)] to our context obtaining

$$(4.21) \quad [E^s, F^r] = \eta \left(\sum_{i=1}^{\min(r,s)} F^{r-i} H_i^{s,r} E^{s-i} \right)$$

where $H_i^{s,r} = (r)_q \cdots (r-i+1)_q \binom{s}{i}_q \prod_{j=1}^{j=i} (K^{-m} - q^{i+j-r-s} K^m)$. We now compute f^2 using (4.21). Since $E^n = 0$ we deduce

$$f^2 = e_\rho H_{n-1}^{n-1, n-1} f = c f$$

where $c = \eta(n-1)_q! \prod_{j=1}^{n-1} (\rho^{-m} - q^{j-n+1} \rho^m)$ and since ρ is exceptional, every factor in c is nonzero, hence $c \neq 0$. Thus $c^{-1}f$ is an idempotent, and the proof is complete. \square

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